Description of dynamic behaviour of three phase asynchronous induction machines by means of space phasors

1 Simplifying assumptions

- Phase symmetric windings are assumed, as well as symmetry of the whole machine structure.
- Only first harmonics (in space) of current coverage, field excitation curve and flux density distribution are taken into account.
- Waveform of all signals is not restricted.
- Resistances and reactances are considered as constant parameters.
- Eddy currents in solid iron as well as iron losses and friction losses are neglected.
- Skin effects are neglected.

2 Systematic of formula symbols

Capital letters	root mean squares instantaneous values dependent on time, not p.u.
Underlined capital letters	complex time phasors
lower-case letters	instantaneous p.u. values dependent on time
underlined lower-case letters	complex space phasors

Used symbols:

f	frequency
ω	angular velocity
t	time
τ	normalized time (p.u.); time constant
s	slip
p	power; number of pole pairs
m	torque
r	ohmic resistance
x	reaktance
u	voltage
i	current
ψ	flux linkage
γ	rotational angle (stator against rotor)
δ	rotational angle (SKS against stator)
φ	phase shift (in time)
θ	moment of inertia (not p.u.)
σ	magnetic leakage factor
κ	magnetic coupling coefficient
β	reciprocal of a time constant

Used indices:

a, b, c	stator phase winding
A, B, C	rotor phase winding
S	stator
R	rotor
N	nominal
(K)	coordinate system, rotating with arbitrary angular velocity
(S)	coordinate system fixed to the stator (SKS)
(R)	coordinate system fixed to the rotor (RKS)
α	real part
β	imaginary part
m	mechanical
el	electrical
L	load
v	losses
μ	magnetizing
h	main
σ	leakage
i	induced
x*	conjugate complex

3 Usage of per unit values

Used reference values are:

- amplitude of stator's nominal voltage (per phase) $\sqrt{2} \cdot U_N$ for all voltages
- amplitude of stator's nominal current (per phase) $\sqrt{2} \cdot I_N$ for all currents
- flux linkage $\sqrt{2} \cdot U_N / \Omega_N$ belonging to nominal voltage and nominal frequency f_N for all flux linkages ($\Omega_N = 2\pi \cdot f_N$)
- impedance U_N / I_N for all impedances
- $1/2\pi$-times the period interval at nominal frequency f_N for time
  Therefore time is measured by an angle $\tau = \Omega_N \cdot t$.
- apparent power $3 \cdot U_N \cdot I_N$ for all powers
- torque derived from nominal apparent power and synchronous angular velocity (at nominal frequency) for all torques.

4 Representation of operation condition of a three phase asynchronous induction machine by means of complex space phasors

Feeding a m-phase symmetrical winding with m alternating currents which are shifted in phase by T/m (T...period interval) generates a rotating current coverage with distinct sinusoidal first harmonic (in space).

Changes in axial direction are neglected, therefore it is possible to describe this sine wave by means of a phasors in the complex plane. The origin is located at the machine's axis of rotation.

When describing the operational behaviour of the machine, three particular angular velocities of the coordinate system are important:

- $\omega_K = 0$ SKS coordinate system fixed to the stator
- $\omega_K = \omega_S$ DKS coordinate system fixed to the rotating magnetic field
- $\omega_K = \omega_m$ RKS coordinate system fixed to the rotor

In a coordinate system fixed to the stator the three phase currents of the stator i_a, i_b, i_c of a three phase winding are united in the stator's current space phasors $\underline{i}_S$.

The corresponding coordinate system is identified by an additional index in brackets:
(S) ... fixed to the stator, (R) ... fixed to the rotor

$$\underline{i}_{S(S)} = \frac{2}{3} \cdot \left(i_a + \underline{a} \cdot i_b + \underline{a}^2 \cdot i_c \right) \qquad (1)$$

$$\text{using } \underline{a} = e^{j\frac{2}{3}\pi} = -\frac{1}{2} + j\frac{\sqrt{3}}{2} \qquad (2)$$

This may also be read as a formal definition.

Solving this equation we obtain the phase currents (using $i_a + i_b + i_c = 0$):

$$i_a = \operatorname{Re}\left(\underline{i}_{S(S)}\right) \tag{3}$$

$$i_b = \operatorname{Re}\left(\underline{a}^2 \cdot \underline{i}_{S(S)}\right) \tag{4}$$

$$i_c = \operatorname{Re}\left(\underline{a} \cdot \underline{i}_{S(S)}\right) \tag{5}$$

Space phasors representing voltage and flux linkage are defined in the same way.

Analogous to that the rotor phase currents are united in the coordinated system fixed to the rotor:

$$\underline{i}_{R(R)} = \frac{2}{3} \cdot \left(i_A + \underline{a} \cdot i_B + \underline{a}^2 \cdot i_C\right) \tag{6}$$

Using a coordinate system rotating with arbitrary angular velocity the conversion ov a space phasors $\underline{i}$ into another coordinate system is shown:

The coordinate system fixed to the stator has the angular velocity $\omega_K = 0$.

The coordinate system fixed to the rotor has the angular velocity $\omega_K = \omega_m$.

The angle of twist γ against the coordinate system fixed to the stator is: $\frac{d\gamma}{d\tau} = \omega_m(\tau)$

An arbitrary coordinate system has the angular velocity ω_K.

The angle of twist δ against the coordinate system fixed to the stator is: $\frac{d\delta}{d\tau} = \omega_K(\tau)$

$\underline{i}_{(K)} = \underline{i}_{(S)} \cdot e^{-j\delta}$ $\quad$ $\underline{i}_{(K)} = \underline{i}_{(R)} \cdot e^{-j(\delta-\gamma)}$ $\quad$ $\underline{i}_{(R)} = \underline{i}_{(S)} \cdot e^{-j\gamma}$

Since angular velocities in general are dependent on time, we have to obey product's rule when differentiating with respect to time:

$$\frac{d\underline{i}_{(S)}}{d\tau} = \frac{d\underline{i}_{(K)}}{d\tau} \cdot e^{j\delta} + \underline{i}_{(K)} \cdot e^{j\delta} \cdot \left(j\frac{d\delta}{d\tau}\right) = \frac{d\underline{i}_{(K)}}{d\tau} \cdot e^{j\delta} + j\omega_K \cdot \underline{i}_{(K)} \cdot e^{j\delta}$$

5 Description of dynamic behaviour of three phase asynchronous induction machines by means of space phasors equations

5.1 Voltage equations

Starting with Maxwell's 2nd law, using a consumer oriented reference system we obtain the voltage equation for one phase of the machine:

$$\mathrm{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \oint \vec{E} \cdot d\vec{s} = -\frac{d\Psi_a}{dt} = R_S \cdot I_a - U_a \quad \Big| \cdot \frac{1}{U_N} \tag{7}$$

$$\Rightarrow \quad u_a = r_S \cdot i_a + \frac{d\psi_a}{d\tau} \quad \Big| \cdot \frac{2}{3} \tag{8}$$

Analogous to that we can write down the voltage equations for mphases b and c:

$$u_b = r_S \cdot i_b + \frac{d\psi_b}{d\tau} \quad \Big| \cdot \frac{2}{3}\underline{a} \tag{9}$$

$$u_c = r_S \cdot i_c + \frac{d\psi_c}{d\tau} \quad \Big| \cdot \frac{2}{3}\underline{a}^2 \tag{10}$$

Multiplying the equations by the indicated factors and summing up equations (8) – (10) yields the stator's voltage equation as space phasors representation:

$$\underline{u}_{S(S)} = r_S \cdot \underline{i}_{S(S)} + \frac{d\underline{\psi}_{S(S)}}{d\tau} \tag{11}$$

Analogous the rotor's voltage equation in the coordinate system fixed to the rotor is:

$$\underline{u}_{R(R)} = r_R \cdot \underline{i}_{R(R)} + \frac{d\underline{\psi}_{R(R)}}{d\tau} \tag{12}$$

Transforming both voltage equations to a common coordinate system yields:

$$\underline{u}_{S(K)} = r_S \cdot \underline{i}_{S(K)} + \frac{d\underline{\psi}_{S(K)}}{d\tau} + j\omega_K \cdot \underline{\psi}_{S(K)} \tag{13}$$

$$\underline{u}_{R(K)} = r_R \cdot \underline{i}_{R(K)} + \frac{d\underline{\psi}_{R(K)}}{d\tau} + j(\omega_K - \omega_m) \cdot \underline{\psi}_{R(K)} \tag{14}$$

The practically important special cases for coordinate system fixed to the stator, the rotor and the rotating magnetic field we may obtain by substituting the special values of angular velocity for ω_K.

5.2 Flux linkage equations

Superposition of the three fundamental field excitation waves generates a total magnetomotive force which generates the main magnetic flux:

$$\underline{i}_{\mu(K)} = \underline{i}_{S(K)} + \underline{i}_{R(K)} \qquad (15)$$

$$\underline{\psi}_h = x_h \cdot \underline{i}_\mu \qquad (16)$$

Furthermore we assume that magnetic leakage fluxes only are dependent on stator respectively rotor current:

$$\underline{\psi}_{S\sigma} = x_{S\sigma} \cdot \underline{i}_S \qquad (17)$$

$$\underline{\psi}_{R\sigma} = x_{R\sigma} \cdot \underline{i}_R \qquad (18)$$

Using that we obtain the equations for total magnetic flux linkage:

$$\underline{\psi}_S = \underline{\psi}_h + \underline{\psi}_{S\sigma} = x_S \cdot \underline{i}_S + x_h \cdot \underline{i}_R \qquad (19)$$

$$\underline{\psi}_R = \underline{\psi}_h + \underline{\psi}_{R\sigma} = x_h \cdot \underline{i}_S + x_R \cdot \underline{i}_R \qquad (20)$$

In this context the following equations apply:

$$x_S = x_h + x_{S\sigma} \qquad (21)$$

$$x_R = x_h + x_{R\sigma} \qquad (22)$$

5.3 Electromagnetic torque; equation of rotational motion

The torque generated by the machine can be derived from a power balance; the electric instantaneous power:

$$P(t) = U_a \cdot I_a + U_b \cdot I_b + U_c \cdot I_c + U_A \cdot I_A + U_B \cdot I_B + U_C \cdot I_C \qquad (23)$$

is divided by the nominal apparent power $S_N = 3 \cdot U_N \cdot I_N$:

$$p(\tau) = \frac{2}{3}\left(u_a \cdot i_a + u_b \cdot i_b + u_c \cdot i_c + u_A \cdot i_A + u_B \cdot i_B + u_C \cdot i_C\right) \qquad (24)$$

Using space phasors the instantaneous power may be written as:

$$p(\tau) = \mathrm{Re}\left(\underline{u}_S \cdot \underline{i}_S^* + \underline{u}_R \cdot \underline{i}_R^*\right) \qquad (25)$$

Instantaneous power may be split into three quantities:

$$p(\tau) = p_v(\tau) + p_\mu(\tau) + p_m(\tau) \qquad (26)$$

$p_v(\tau)$ ... losses in stator and rotor
$p_\mu(\tau)$ ... change of stored magnetic energy
$p_m(\tau)$ ... mechanic power

Using the voltage equations we obtain:

$$p(\tau) = \mathrm{Re}\left(r_S \cdot \underline{i}_S \cdot \underline{i}_S^* + r_S \cdot \underline{i}_R \cdot \underline{i}_R^*\right) + \mathrm{Re}\left(\frac{d\underline{\psi}_S}{d\tau} \cdot \underline{i}_S^* + \frac{d\underline{\psi}_R}{d\tau} \cdot \underline{i}_R^*\right)$$
$$\ldots\ldots\ldots + \mathrm{Re}\left(j\omega_K \cdot \left(\underline{\psi}_S \cdot \underline{i}_S^* + \underline{\psi}_R \cdot \underline{i}_R^*\right)\right) - \mathrm{Re}\left(j\omega_m \cdot \underline{\psi}_R \cdot \underline{i}_R^*\right) \quad (27)$$

Transforming the 3rd term we find:

$$\mathrm{Re}\left(j\omega_K \cdot \left(\underline{\psi}_S \cdot \underline{i}_S^* + \underline{\psi}_R \cdot \underline{i}_R^*\right)\right) = \mathrm{Re}\left(j\omega_K \cdot \underline{\Gamma}\right)$$

$\underline{\Gamma} = \left(x_S \cdot \underline{i}_S + x_h \cdot \underline{i}_R\right) \cdot \underline{i}_S^* + \left(x_h \cdot \underline{i}_S + x_R \cdot \underline{i}_R\right) \cdot \underline{i}_R^* = \quad \Rightarrow \underline{\Gamma}$ is a real number. $\Rightarrow \mathrm{Re}\left(j\omega_K \cdot \underline{\Gamma}\right) = 0$

$$\ldots\ldots x_S \cdot \underline{i}_S \cdot \underline{i}_S^* + x_R \cdot \underline{i}_R \cdot \underline{i}_R^* + x_h \cdot \left(\underline{i}_R \cdot \underline{i}_S^* + \left(\underline{i}_R \cdot \underline{i}_S^*\right)^*\right)$$

Comparing both expressions of instantaneous power yields:

$$p_v(\tau) = \mathrm{Re}\left(r_S \cdot \underline{i}_S \cdot \underline{i}_S^* + r_S \cdot \underline{i}_R \cdot \underline{i}_R^*\right)$$
$$p_\mu(\tau) = \mathrm{Re}\left(\frac{d\underline{\psi}_S}{d\tau} \cdot \underline{i}_S^* + \frac{d\underline{\psi}_R}{d\tau} \cdot \underline{i}_R^*\right) \quad (28)$$
$$p_m(\tau) = -\mathrm{Re}\left(j\omega_m \cdot \underline{\psi}_R \cdot \underline{i}_R^*\right)$$

Mechanical power can be formulated by using stator values:

$$p_m(\tau) = -\mathrm{Re}\left(j\omega_m \cdot \left(x_h \cdot \underline{i}_S + x_R \cdot \underline{i}_R\right) \cdot \underline{i}_R^*\right) \quad (29)$$

Since $\underline{i}_R \cdot \underline{i}_R^*$ is a real number, the follwing $\mathrm{Re}\left(j\omega_m \cdot x_R \cdot \underline{i}_R \cdot \underline{i}_R^*\right) = 0$ is valied and from there:

$$p_m(\tau) = -\mathrm{Re}\left(j\omega_m \cdot x_h \cdot \underline{i}_S \cdot \underline{i}_R^*\right) \quad (30)$$

Since $\underline{i}_S \cdot \underline{i}_S^*$ is a real number, we may add $x_S \cdot \underline{i}_S^*$ to $x_h \cdot \underline{i}_R^*$:

$$p_m(\tau) = -\mathrm{Re}\left(j\omega_m \cdot \underline{i}_S \cdot \underline{\psi}_S^*\right) = \mathrm{Im}\left(\omega_m \cdot \underline{i}_S \cdot \underline{\psi}_S^*\right) = \omega_m \cdot m_{el} \quad (31)$$

$$\Rightarrow m_{el} = \mathrm{Im}\left(\underline{i}_S \cdot \underline{\psi}_S^*\right) \quad (32)$$

Using reference values of torque and time we obtain the equation of rotational motion using per unit values:

$$\Theta \cdot \frac{d\Omega_m}{dt} = M_{el} - M_L \quad (33)$$

$$M_{S,N} = \frac{3 \cdot U_N \cdot U_N}{\Omega_S / p} \qquad \tau_m = \Omega_N \cdot T_M = \Omega_N \cdot \Theta \cdot \frac{\Omega_{m,N}}{M_{S,N}} \quad (34)$$

$$\tau_m \cdot \frac{d\omega_m}{d\tau} = m_{el} - m_L \quad (35)$$

5.4 The complete system of equations

Using a coordinate system rotating with arbitrary angular velocity ω_K the machine's equations read as follows:

$$\begin{aligned}
\underline{u}_S &= r_S \cdot \underline{i}_S + \frac{d\underline{\psi}_S}{d\tau} + j\omega_K \cdot \underline{\psi}_S \\
\underline{u}_R &= r_R \cdot \underline{i}_R + \frac{d\underline{\psi}_R}{d\tau} + j(\omega_K - \omega_m) \cdot \underline{\psi}_R \\
\underline{\psi}_S &= x_S \cdot \underline{i}_S + x_h \cdot \underline{i}_R \\
\underline{\psi}_R &= x_h \cdot \underline{i}_S + x_R \cdot \underline{i}_R \\
\tau_m \cdot \frac{d\omega_m}{d\tau} &= m_{el} - m_L \\
m_{el} &= \mathrm{Im}\left(\underline{i}_S \cdot \underline{\psi}_S^*\right)
\end{aligned} \qquad (36)$$

To obtain currents from flux linkages, we have to invert equations of flux linkage:

$$\begin{aligned}
\underline{i}_S &= +\frac{1}{\sigma \cdot x_S} \cdot \underline{\psi}_S - \frac{\kappa}{\sigma \cdot x_h} \cdot \underline{\psi}_R \\
\underline{i}_R &= -\frac{\kappa}{\sigma \cdot x_h} \cdot \underline{\psi}_S + \frac{1}{\sigma \cdot x_R} \cdot \underline{\psi}_R
\end{aligned} \qquad (37)$$

The total magnetic leakage factor is defined as follows:

$$\sigma = 1 - \kappa = 1 - \frac{x_h^2}{x_S \cdot x_R} \qquad (38)$$

6 The zero-sequence system

Generally the feeding system of alternating currents is no longer symmetric as assumed in chapter 4, the sum of the currents is no longer = 0:

$$i_0 = \frac{1}{3} \cdot (i_a + i_b + i_c) \qquad (39)$$

The zero-sequence system makes no contribution to the space phasor:

$$i_0 + \underline{a} \cdot i_0 + \underline{a}^2 \cdot i_0 = 0 \qquad (40)$$

The zero-sequence system has no effect on the main field and causes only parasitic effects. Therefore it has to be calculated separately, it has to be added during back transformation:

$$u_0 = r_S \cdot i_0 + \frac{d\psi_0}{d\tau} \qquad (41)$$

$$i_a = i_0 + \mathrm{Re}\left(\underline{i}_{S(S)}\right) \qquad (42)$$